

# Buckling Cascade of Thin Plates: Forms, Constraints and Similarity

B.Roman\* and A.Pocheau†

IRPHE, UMR 6594 CNRS, Universités Aix-Marseille I & II  
Centre de Saint-Jérôme, S.252, 13397 Marseille Cedex 20, France

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## Abstract

We experimentally study compression of thin plates in rectangular boxes with variable height. A cascade of buckling is generated. It gives rise to a self-similar evolution of elastic reaction of plates with box height which surprisingly exhibits repetitive vanishing and negative stiffness. These features are understood from properties of Euler's equation for elastica.

Elastic thin plates submitted to a sufficiently large in-plane load are well-known to spontaneously bend due to buckling instability [1]. However, this phenomenon, which is often quoted as a canonical example of primary bifurcation in out-of-equilibrium systems, has been little investigated in the fully non-linear regime [2, 3, 4, 5, 6]. Yet, it provides an appealing opportunity of yielding pattern formations within a non-local variational framework - elasticity - free of significant noise disturbances. In addition, it plays an essential role in the mechanical resistance of homogenous or layered materials and involves important practical implications regarding packaging, safety structures or in-load behaviour of plywood or film-substrate composites. Therefore, for both fundamental and practical reasons, the non-linear buckling regime of thin plates warrants a renewal of interest from physicists.

This Letter is devoted to experimentally studying buckling of thin plates from the quasi-linear regime to the far non-linear regime. In practice, plates are confined in a box involving *fixed* horizontal boundaries enclosing an area *smaller* than those of plates, and a *variable* height  $Y$  (Fig.1). Plates are thus forced to bend with a bending amplitude imposed by the box height. In particular, *iterated* buckling instabilities can then be triggered by simply reducing box height. Doing this, buckled plates display usual geometry of out-of-equilibrium patterns: large patches of slightly curved folds separated by sharp defects where stretch is mainly localized.

We choose here to put attention on the large patches where bending seems dominant. To this aim, we model them by parallel folds, i.e. by *unidirectional* buckling. This is achieved by taking rectangular plates and a rectangular box involving a *constant* length  $X$ , *smaller* than those of plates. This way, original mechanical behaviours have been shown in a framework simple enough to be easily handled. In particular, the occurrence and the nature of the instability *cascade* have been simply understood by using only mechanical invariants and similarity properties.

Reaction force  $F$  of buckled plates on the top and bottom boundaries of the confining box shows interesting non-linear features: a puzzling repetitive *vanishing* of  $F$  as box height  $Y$  is reduced and,

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\*e-mail: roman@lrc.univ-mrs.fr

†e-mail: pocheau@lrc.univ-mrs.fr

in some states, a surprising *negative* stiffness  $-dF/dY < 0$ . Both emphasize the practical differences between the present system and common springs. More generally, the relation  $F(Y)$  is shown mainly to exhibit self-similar behaviour. Self-similarity is stressed by mapping most of the cascade of buckling bifurcations onto a definite curve with simple rescaling. These non-linear properties are shown to correspond to those of Euler's equation for elastica.

*Experimental set-up:* A thin rectangular sheet with dimensions  $l \times L$  ( $l < L$ ) and thickness  $h$  is clamped on its smallest sides to a bottom plate so as to display an initial bend of height  $Y_1$  (Fig.1). This is obtained by placing the clamped ends at a distance  $X$  shorter than the sheet length  $L$  :  $X = 22\text{cm}$ ,  $L = 23.3\text{cm}$ ,  $L/X = 1.06$ ,  $l = 10.1\text{cm}$ ,  $Y_1 = 35\text{mm}$ . The bottom plate is pushed by a pneumatic jack towards a fixed parallel upper glass plate. Parallelism between plates during translation is ensured by three guiding axes. As the distance between confining plates shrinks, the bent sheet comes in contact with the upper plate and starts compressing. Its compression is stopped by three stepping motors which prevent the bottom plate from moving closer to the upper plate than a controlled distance  $Y$ .

At this stage, the buckled plate is thus constrained into a rectangular box still with a lateral extent  $X$  but a reduced height  $Y \leq Y_1$ . Note that the same elastic state could also have been obtained by keeping the height fixed at an initial value  $Y$  and reducing the lateral box size to  $X$  starting from above [4, 6, 7]. However, height reduction has been found more convenient here than width reduction to accurately scan a large part of the instability cascade while keeping within the elastic regime. Moreover, in contrast with the configuration studied in [4], plates are not held on two opposite sides so as to avoid two-dimensional folds [2]. On the other hand, the out-plane  $Y$ -component of the elastical reaction of sheets on the box has been studied instead of the usual in-plane  $X$ -component [4, 6, 7]. This is because the former proved to be more sensitive to buckling bifurcation.

Height values  $Y$  are deduced from stepping motor displacements. Forces applied to the jack and to each of the three motors are measured by four load gauges. The difference  $F$  between the former and the latter then gives the normal mechanical response of the sheet. Sheet forms are observed by light diffusion from the side (Fig.2) or by light reflexion from the top (Fig.3) in which case contacts with either plate appear as bright domains.

*Elastic forms:* All forms described below, either buckled or not, are *steady*. Compressing the sheet below the height  $Y_1$  first gives rise to a contact line with the upper plate (Fig.2a). At a critical value  $Y_1^p$ , the contact domain becomes a plane through a continuous transition. A measurable part of the sheet is then flat and in contact with the upper plate (Fig.2b). Still reducing  $Y$ , this flat part extends and eventually buckles at another threshold  $Y_1^b$ . A new fold is then formed but with a height too small to make contact with the other plate (Fig.2c). Since no external force is applied at its central point (here, the bottom of the fold), we call it a free-standing fold. As  $Y$  is further reduced, this fold gets closer to the other plate and eventually touches it at a value  $Y_2$  of  $Y$  (Fig.2d). The sheet then

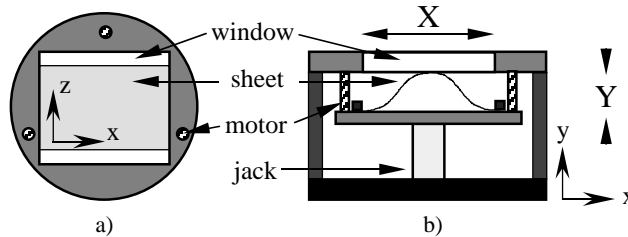


Figure 1: Sketch of the experimental set-up. a) Top view b) Side view.

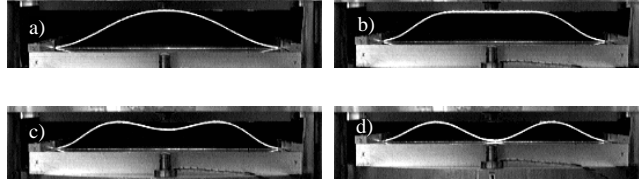


Figure 2: Transition from one to two folds: side view of polycarbonate sheet,  $h = 1mm$ ,  $E \approx 2GPa$ . *Steady* forms a) Contact line:  $Y_1 > Y > Y_1^p$ , b) Contact plane:  $Y_1^p > Y > Y_1^b$ , c) Post-buckling with free-standing fold:  $Y_1^b > Y > Y_2$ , d) Two folds similar to the one of Fig.2a:  $Y_2 > Y > Y_2^p$ .

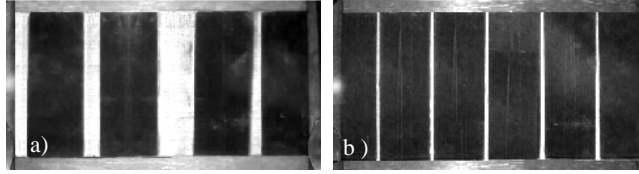


Figure 3: Multi-folds: top view of steel sheet,  $E \approx 200GPa$ . a)  $h = 0.3mm$ , contact planes with two folds, b)  $h = 0.1mm$ , contact lines with three folds.

exhibits two similar folds in contact with both compressing plates.

When further reducing  $Y$ , the same evolution resumes: all contact domains flatten (Fig.3a) until *one* of them, the *largest*, buckles, thereby creating a new fold which eventually reaches the other plate at some still lower box height  $Y_3$  (Fig.3b). This way, an increasing number  $n$  of folds connecting the upper and bottom plates are created in cascade, at box heights  $Y_n$ , all by the same procedure.

*Elastic forces:* Figure 4 shows, for a typical experiment, a plot of the reduced force  $F/F(Y_1^b)$  brought about by a sheet on the set-up plates versus the imposed reduced height  $Y/Y_1$ . Interestingly,  $F$  *vanishes* at several definite box heights. On the way to such states, the sheet therefore displays elastic forces  $F$  *decreasing* with the allowed height  $Y$ :  $dF/dY > 0$ . It thus reacts with a larger force when expanded and a smaller force when compressed, thus showing an original elastic behaviour actually opposite to that of springs. In particular, owing to the unusual negative stiffness  $-dF/dY < 0$ , the sheet should spontaneously collapse under the pressure of the pneumatic jack until a new equilibrium is reached, thereby giving rise to a loss of height control stability. Such instability of the control parameter  $Y$  is fortunately inhibited here thanks to the reaction of the spacing motors. A similar decrease of constraint with height has been observed in the somewhat different context of curved strips [8] but only in the weakly non-linear regime.

The remainder of the letter is devoted to understanding the origin of negative stiffness together with the main properties of the buckling cascade. This will be done within Euler's equation for thin plates.

*Elastica and similarity analysis:* Figure 3 shows a translational invariance of sheet form along the  $z$ -axis [9]. This property results from the translational invariance of the initial bent state and the homogeneity of box height reduction. Following it, sheet geometry reduces to that of a curved line: the sheet cut by the  $(x,y)$  plane. We shall parametrize this line by its curvilinear abscissa  $s$  and the angle  $\theta$  between its tangent and the  $x$ -axis. On the other hand, the small thickness of the sheet ( $h/L < 1.5 \cdot 10^{-3}$ ) implies that its flexural rigidity is very weak compared with its extensional rigidity.

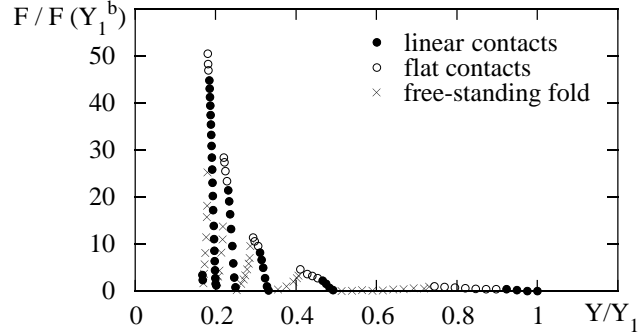


Figure 4: Reduced reaction force  $F/F(Y_1^b)$  on the plates versus reduced fold amplitude  $Y/Y_1$ . Steel sheets:  $h = 0.1mm$ ,  $E \approx 200Gpa$ ,  $Y_1 = 35mm$ ,  $F(Y_1^b) = 6.06N$ . Sequence yielding six folds. Linear contacts ( $\bullet$ ), flat contacts ( $\circ$ ), free-standing folds ( $\times$ ).

Sheet is then expected to accomodate height reduction much more by curvature than by extension, except at singularities [3] which will not appear here.

The elasticity problem is thus reduced to an inextensible flexible uniform strip subjected to external loads. Its equation has been put forward by Euler by expressing mechanical equilibrium of strip elements (1). Neglecting gravity effects [10], force equilibrium shows that the tension force  $\mathbf{T}$  sustained by strip elements is a constant between contacts with external systems (here top and bottom plates). Provided that sheet curvature radius  $R$  is large with respect to sheet thickness  $h$ , moments applied to strip elements can be expanded at first order in  $1/R$ : this corresponds to Hooke's regime. Then, moment equilibrium yields Euler's equation of elastica [1]:

$$EI \frac{d^2\theta}{ds^2} = -p \sin(\theta) + q \cos(\theta) \quad (1)$$

Here  $E$  is the Young's modulus of the material,  $I = lh^3/12$  the inertia moment of a sheet section and  $p$  ( $q$ ) the  $x$  ( $y$ )-component of tension force  $\mathbf{T}$ . Boundary conditions are  $\theta = 0$  at clamped ends. Global constraints are imposed by the prescribed sheet length  $L$  and the box dimensions  $(X, Y)$ . The great interest of this equation, as opposed to other elasticity equations, is that it is equally valid for large distortions, provided that sheet curvature radius remains large with respect to sheet thickness.

Our purpose will not be to solve Euler's equation directly but, instead, to link its solutions by similarity relations. On a primary ground, dimensional analysis only states that reaction forces on horizontal plates,  $F$ , and lateral sides,  $G$ , are linked to height  $Y$  and sheet length  $L$  by a relationship at fixed  $X/L$ :  $FL^2/EI = \phi(GL^2/EI, Y/L)$ . Exploiting properties of elastica, we shall show that, on specific states,  $\phi(.,.)$  reduces to a monovariate function  $\psi(.)$  of variable  $Y/L$  exhibiting discrete scaling invariance:  $\psi(x/n) = n^3\psi(x)$ ,  $n \in N$ . This will be obtained in two steps: first by building, from a given solution in a given box, other solutions in boxes of larger width but same height; second, by mapping them to solutions pertaining to a box involving the initial width but a smaller height.

Euler's equation (1) applies to free branches between the top and bottom plates. These branches are connected by contacts with plates which will be assumed frictionless here. Accordingly, plates act on the sheet by forces only directed in the  $y$ -direction and with no moment. In this case, mechanical equilibrium implies that both  $p$  and  $d\theta/ds$  are continuous at the contact points. Moreover, as  $p$  is a constant of each free branch, it is therefore *invariant* all along the sheet.

These properties enable us to build a solution in larger boxes by replication. Consider a symmetric fold as displayed in the experiment (Fig.1a). Because of clamping boundary conditions ( $\theta = 0$ ) and

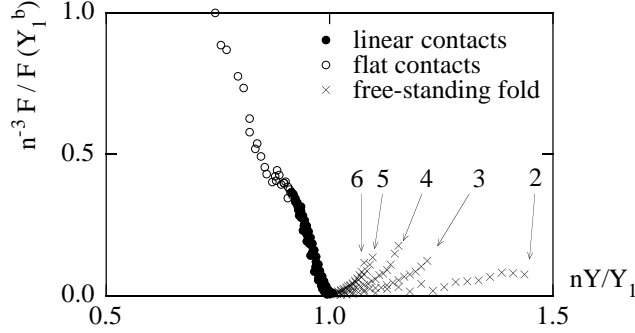


Figure 5: Rescaling of the bifurcation diagram of Fig.4 by scaling transformation (2). Here,  $n$  is the total number of sheet folds, including free-standing folds. It is indicated for states involving free-standing folds because of lack of dynamic similarity, but would be superfluous for remaining states since scaling (2) is satisfied.

fold symmetry ( $d\theta/ds(s) = d\theta/ds(L-s)$ ), both  $\theta$  and  $d\theta/ds$  take the *same* values at sheet ends. On the other hand, Euler's equation (1) is *autonomous* since all its coefficients are constant on each free branch of the sheet. Thus, pursuing the symmetric fold by itself actually provides a relevant solution for elastica. This means that the curve made by two twin symmetric folds indeed corresponds to the form of a compressed sheet. By iterating this procedure, one then gets an elastic solution for a sheet of length  $nL$  in a box of width  $nX$  and height  $Y$  in term of a chain of  $n$  symmetrical folds of length  $L$  compressed within boxes of width  $X$  and height  $Y$ .

We now use the fact that, as the sheet is thin and its geometry smooth, its thickness is not a relevant lengthscale of the elastic problem [11]. The system thus involves no characteristic scale and is therefore scale-invariant. Accordingly, zooming a shape gives a relevant shape for another sheet in another box. In particular, zooming out the previous solution with  $n$  folds by a factor  $1/n$  gives a  $n$ -folds solution for a sheet of length  $L$  compressed in a box of width  $X$  and height  $Y/n$ .

Given a symmetric elastic state for an allowed height  $Y$ , we have thus built a family of elastic states for a series of allowed height  $Y/n$ . This result enables us to connect the physical properties pertaining to the stable branches of solutions generated by successive buckling bifurcations *without* solving elastica in detail. In particular, the force  $F$  acting on the sheet can now be easily derived by following its changes during the above transformations.

By definition, fold replication does not change the value of the sheet tension  $(p, q)$ , since each fold is the same and Euler's equation (1) is autonomous. On the other hand, zooming conserves the angle  $\theta$  but changes the curvilinear abscissa  $s$  into  $s/n$ . However, according to (1), this corresponds equivalently to changing  $p$  into  $n^2p$  and  $q$  into  $n^2q$ . This 'dynamic similarity' means that the solution of length  $L$  in a box of height  $Y/n$  and width  $X$  is relevant to a tension  $(n^2p, n^2q)$ . Finally, as each of the  $n$  folds exerts a force  $2q$  on the upper plate ( $q$  for each contact), one obtains the scaling transformation:

$$\begin{aligned} Y &\rightarrow n^{-1}Y \\ F &\rightarrow n^3F \end{aligned} \quad (2)$$

A direct consequence of this scaling is that force  $F$  goes to zero for every  $Y_n = Y_1/n$ , since  $F$  actually vanishes when the upper plate just touches the sheet ( $Y = Y_1$ , Fig.1a). Accordingly, for  $Y = Y_n$ , sheet states are the same as if there were *no contact* with the upper plate [12].

In the way scaling (2) has been derived, it applies to a family of sheets, each made of a series of identical symmetric folds, the folds of different sheets being linked by similarity. As may be directly

noticed on Figs.2a,2d,3b and quantitatively confirmed by rescaling and superposition, these conditions are indeed satisfied on states involving neither planar contact nor free-standing folds. Hereafter, they will be referred to as ‘linear contacts’. To exhibit the expected scaling property (2), we plotted rescaled reduced force  $n^{-3}F/F(Y_1^b)$  versus rescaled reduced height  $nY/Y_1$  on Fig.5 for the same data as those of Fig.4. Excellent collapse is obtained for linear contacts ( $\bullet$ ), thereby demonstrating the scaling.

Collapse on Fig.5 even extends to states involving planar contacts with different lengths ( $\circ$ ) (Fig.2b,3a). This may be surprising since sheets are then *no longer* made of a series of identical symmetrical folds, as assumed in deriving (2), so that there is at this stage *no longer* reason for similarity to apply. However, we show below that, regarding the force  $F$ , these states may be equally considered to be made of such a series of identical folds. Accordingly, their reaction force still follow similarity despite their actual form does not.

When a sheet exhibits a planar contact region, it satisfies  $\theta = 0$  and  $d\theta/ds = 0$  at the contact points. On the other hand, due to integrability of Euler’s equation (1), the quantity  $H = EI(d\theta/ds)^2 + p\cos(\theta) + q\sin(\theta)$  is conserved along each free branch. Accordingly, the next contact points found by following the sheet starting from the contact plane also satisfy  $d\theta/ds = 0$ . By iteration this property extends from fold to fold to *any* contact point of the sheet. In other words, contacts must be either planar or linear *altogether* [13]. On the other hand, planar contacts correspond to ‘rest’ states of the dynamic system (1). As equation (1) is autonomous, their length is thus of no importance for the subsequent motion (i.e. for the form of the next sheet fold). Accordingly, as long as the total length of planar domains is conserved, one may change the planar domain distribution *without* changing the shape of the free branches and, therefore, the force they exert on plates. Note also that modification of planar regions do not change the force  $F$  by itself since these flat parts correspond to  $q = 0$  anyway. Accordingly, for any sheet displaying a flat contact domain, there exists another elastic state made of identical symmetric folds and referring to the same force  $F$ . As these states satisfy scaling (2), this explains that collapse in Fig.5 extends to states involving planar contacts.

Whereas the distribution of flat parts has no influence on the reaction force  $F$ , we stress that it parametrizes buckling’s occurrence. This comes from the fact that, as the  $x$ -component  $p$  of sheet tension is a constant, the buckling threshold of flat parts at given  $(X, Y, L)$  solely depends on their size. Thus, the larger a flat part, the larger the height value where buckling occurs. To reduce this dependency on sheet form, only sheets involving a *single* flat part have been considered in Figs.4 and 5.

In Fig.5, states involving free-standing fold ( $\times$ ) show no collapse. Absence of collapse is due to the fact that a free-standing fold cannot be distributed along the sheet, thereby making scaling (2) inapplicable. On the other hand, branches exhibit a discontinuity with the branch relevant to planar contacts. This simply comes from the step of  $n$  at each buckling’s occurrence [14].

*Conclusion:* Weakly curved patches of buckled thin plates have been modeled by unidirectional buckling for which stretch actually vanishes. Such sheets have been studied in the *far* non-linear domain where many buckling bifurcations have already occurred. Then, major elastic features such as negative stiffness or vanishing of the normal constraint applied to compressing plates can be simply explained from similarity considerations and intrinsic properties of Euler’s equation. In particular, most of the bifurcation diagram is shown to collapse on a curve, thereby stressing the generation of a cascade of similar sub-scales as the bifurcation diagram is explored.

On a more general ground, thin plates compressed in rectangular boxes provide an elasticity example of cascade generation of self-similar scales in out-of-equilibrium systems. In particular, the series of instabilities encountered away from equilibrium appear all similar to the primary instability. Self-similarity is characterized here by a multiplicative periodicity which includes, but goes beyond,

log-periodicity.

These results, which are particularly relevant to layered composites, show that compressed plates on two perpendicular directions alternately turn from a *fragile* to a *robust* state as their confinement is increased. More generally, they show a way of handling elasticity without looking directly for elastic solutions. Finally, they provide a useful basis for addressing the more complex case of sheets displaying folds in all directions.

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- [9] Careful inspection of Fig.3a shows a slight width variation of flat parts on the  $z$ -direction. This follows from the fact that distribution of flat parts is a degree of freedom (zero mode) of Euler's equation (1).
- [10] Ratio of gravitational ( $\rho ghLLY$ ) to elastic ( $EILY^2$ ) energy is at most  $10^{-2}$  here.
- [11] Sheet thickness cannot be considered as zero, otherwise the factor  $EI$  would vanish in (1). In this sense, it stands as a singular perturbation here.
- [12] Note that this plate is nevertheless necessary to sustain folds since, otherwise, all folds would be unstable but the one which is clamped at both ends ( $Y = Y_1$ , Fig.1a).
- [13] This includes flat domains reduced to a line in so far as  $d\theta/ds = 0$  at the contact points.
- [14] Although discrete change of  $n$  is necessary here, locating it at the beginning or at the end of buckling is arbitrary: it turns out considering a free-standing fold as an actual fold or not.